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NONLINEAR DEFORMATION AND STABILITY OF REINFORCED ELLIPTICAL CYLINDRICAL SHELLS LOADED BY INTERNAL PRESSURE UNDER TWISTING AND BENDING

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The problem of stability of cylindrical shells with an elliptical cross-sectional contour reinforced by a set of stringers under combined loading by bending and twisting moments, transverse force, and internal pressure is studied with the use of the variational method of finite elements in displacements. The subcritical stress–strain state of the shells is assumed to be moment and nonlinear. The effect of nonlinearity of deformation of the shells and their ellipticity on the critical loads and buckling type is determined.

Key words: *elliptical cylindrical shells, twisting, bending, internal pressure, nonlinear deformation, stability, method of finite elements.*

Introduction. Projects of leak-proof cabins of airplane fuselages with oval and elliptical cross sections have been recently developed. Such cabins allow more effective use of the inner space of fuselages for accommodation of passengers. Buckling of the cabin skin is inadmissible, because multiple repetition of buckling can lead to formation of cracks and, as a consequence, to cabin decompression. The global loss of stability (together with reinforcing elements) leads to failure of the entire structure.

Stability of non-circular shells, as compared to circular shells, has not been adequately studied yet. In most publications on this topic, the problems of shell stability were solved in the first approximation with the use of the linear theory of shells and under the assumptions of the moment-free subcritical stress state, etc. The error of the linear theory was not estimated. In some cases, this theory is not sufficiently reliable, because the fuselage shells are thin-walled structures and usually exhibit nonlinear deformation under large displacements.

Displacements of Finite Elements of Noncircular Cylindrical Shells as Solid Bodies. In the case of displacements of finite elements (FEs) as solids, the strain components are equal to zero. Assuming that the linear strain components, changes in curvatures, and twisting are also equal to zero [1], we obtain

$$\varepsilon_1 = u_x = 0, \qquad \varepsilon_2 = k_2(v_\beta + w) = 0, \qquad \varepsilon_3 = v_x + k_2 u_\beta = 0,$$
$$\chi_1 = w_{xx} = 0, \qquad \chi_2 = k_2[k_2(v - w_\beta)]_\beta = 0, \qquad \chi_3 = [k_2(v - w_\beta)]_x = 0. \tag{1}$$

Here, u, v, and w are the tangential and normal displacements, R and $k_2 = R^{-1}$ are the radius and curvature of the cross-sectional contour, β is the angle between the normal to the cross-sectional contour and the semi-axis b of the cross section, and x is the longitudinal coordinate (Fig. 1). The subscripts x and β indicate differentiation with respect to the corresponding variables.

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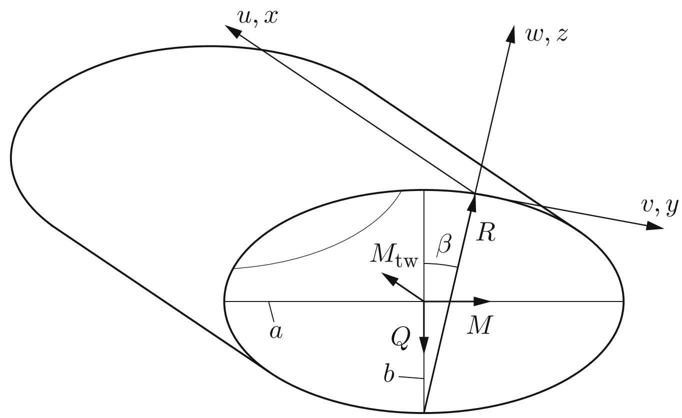

Fig. 1. Schematic of the shell.

Integrating Eqs. (1), we obtain the functions of displacements of FEs as solids:

$$u = C_1\psi_1 + C_2\psi_2 + C_6, \qquad v = C_3c + C_4s - C_5(\psi_1 c + \psi_2 s) + (C_2c - C_1s)x,$$
$$w = C_3s - C_4c - C_5(\psi_1 s - \psi_2 c) + (C_2s + C_1c)x, \tag{2}$$
$$\psi_1 = \int Rs\,d\beta, \qquad \psi_2 = -\int Rc\,d\beta, \qquad c = \cos\beta, \qquad s = \sin\beta$$

(C_i are arbitrary constants).

In the case of an elliptical shell, we have the relations

$$\frac{z^2}{b^2} + \frac{y^2}{a^2} = 1, \quad R = \frac{a^2b^2}{d^3}, \quad d^2 = a^2s^2 + b^2c^2, \quad k_2 = \frac{d^3}{a^2b^2}, \quad \psi_1 = -\frac{b^2c}{d}, \quad \psi_2 = -\frac{a^2s}{d},$$

and the relations for a circular shell are

$$z^2 + y^2 = R^2, \qquad \psi_1 = -Rc, \qquad \psi_2 = -Rs.$$

Finite Element and Algorithm of Solving the Problem. Let us divide the shell by the lines of the principal curvatures into m parts over the generatrix and into n parts over the guide line. Thus, the shell is presented as a set of $m \times n$ curvilinear rectangular FEs. Using bilinear approximation of deformational tangential displacements and bicubic approximation for normal displacements, we obtain the following expressions for the total displacements of natural curvature FE points with allowance for Eqs. (2):

$$u = a_1xy + a_2x + a_3y + a_4 + a_6\psi_2 + a_{20}\psi_1,$$
$$v = a_5xy + a_6xc + a_7y + a_8(\psi_1c + \psi_2s) - a_{20}xs + a_{23}c - a_{24}s, \tag{3}$$
$$w = a_9x^3y^3 + a_{10}x^3y^2 + a_{11}x^3y + a_{12}x^3 + a_{13}x^2y^3 + a_{14}x^2y^2 + a_{15}x^2y + a_{16}x^2 + a_{17}xy^3$$
$$+ a_{18}xy^2 + a_{19}xy + a_{20}xc + a_{21}y^3 + a_{22}y^2 + a_{23}s + a_{24}c + a_6xs + a_8(\psi_1s - \psi_2c).$$

System (3) can also be written in the matrix form as

$$\tilde{\boldsymbol{u}} = P\boldsymbol{a}, \tag{4}$$

where $\tilde{\boldsymbol{u}} = \{u, v, w\}^{\mathrm{t}}$ is the vector of displacements of the FE mid-surface points, $\boldsymbol{a} = \{a_1, \ldots, a_{24}\}^{\mathrm{t}}$ is the vector of unknown coefficients a_i of the polynomials, and P is the 3×24 matrix of coupling whose elements are multipliers at the coefficients a_i in Eqs. (3). Expressing the coefficients a_i via the node unknowns, we obtain

$$\boldsymbol{a} = B^{-1}\bar{\boldsymbol{u}}, \tag{5}$$

where $\bar{\boldsymbol{u}} = \{u_i, v_i, w_i, \vartheta_{1i}, \vartheta_{2i}, w_{xyi}, u_j, v_j, w_j, \vartheta_{1j}, \vartheta_{2j}, w_{xyj}, u_k, \ldots, w_{xyk}, u_n, \ldots, w_{xyn}\}^{\mathrm{t}}$ is the vector of node displacements, angles of turning, and mixed derivatives of FE deflection; B is the 24×24 matrix whose non-zero elements have the form

$$b_{1j} = p_{1j}, \quad b_{2j} = p_{2j}, \quad b_{3j} = p_{3j}, \quad b_{4j} = (p_{3j})_x, \quad b_{5j} = (p_{2j} - (p_{3j})_y)/R,$$
$$b_{6j} = (p_{3j})_{xy}, \quad x = -a_1, \quad y = -b_1, \quad b_{7j} = p_{1j}, \quad b_{8j} = p_{2j}, \quad b_{9j} = p_{3j}, \quad b_{10j} = (p_{3j})_x,$$
$$b_{11j} = (p_{2j} - (p_{3j})_y)/R, \quad b_{12j} = (p_{3j})_{x\beta}, \quad x = -a_1, \quad y = b_1, \quad b_{13j} = p_{1j}, \quad b_{14j} = p_{2j},$$
$$b_{15j} = p_{3j}, \quad b_{16j} = (p_{3j})_x, \quad b_{17j} = (p_{2j} - (p_{3j})_\beta)/R, \quad b_{18j} = (p_{3j})_{xy}, \quad x = a_1, \quad y = -b_1,$$
$$b_{19j} = p_{1j}, \quad b_{20j} = p_{2j}, \quad b_{21j} = p_{3j}, \quad b_{22j} = (p_{3j})_x, \quad b_{23j} = (p_{2j} - (p_{3j})_y)/R,$$
$$b_{24j} = (p_{3j})_{xy}, \quad x = a_1, \quad y = b_1 \quad (j = 1, \ldots, 24, \quad a_1 = L/(2m), \quad b_1 = l/(2n)).$$

Here, L and l are the characteristic sizes of the shell along the generatrix and the guide line, respectively. Each FE node has six unknowns; therefore, the FE has 24 degrees of freedom.

Substituting Eq. (5) into Eq. (4), we obtain the dependence of displacements of FE points on the node unknowns:

$$\tilde{\boldsymbol{u}} = PB^{-1}\bar{\boldsymbol{u}}.$$

Let us write the nonlinear Cauchy relations for strains and changes in curvatures of the shell mid-surface [1]

$$\boldsymbol{e} = \boldsymbol{e}_l + \boldsymbol{e}_n, \tag{6}$$

where $\boldsymbol{e}_l = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \chi_1, \chi_2, \chi_3\}^{\mathrm{t}}$ is the vector with the linear components

$$\begin{gathered} \varepsilon_1 = u_x, \qquad \varepsilon_2 = k_2(v_\beta + w), \qquad \varepsilon_3 = v_x + k_2 u_\beta, \\ \chi_1 = -w_{xx}, \qquad \chi_2 = k_2(k_2(v - w_\beta))_\beta, \qquad \chi_3 = k_2(v - w_\beta)_x, \end{gathered} \tag{7}$$

$\boldsymbol{e}_n = \{\bar{\boldsymbol{e}}_n^{\mathrm{t}}, 0, 0, 0\}^{\mathrm{t}}$ ($\bar{e}_n = \{\varepsilon_{1n}, \varepsilon_{2n}, \varepsilon_{3n}\}^{\mathrm{t}}$) is the vector with the nonlinear components

$$\varepsilon_{1n} = \vartheta_1^2/2 = (w_x)^2/2, \quad \varepsilon_{2n} = \vartheta_2^2/2 = k_2^2(v - w_\beta)^2/2, \quad \varepsilon_{3n} = \vartheta_1\vartheta_2 = -k_2 w_x (v - w_\beta). \tag{8}$$

According to [1], the law of elasticity for the shell has the form

$$\boldsymbol{T} = D\boldsymbol{e}. \tag{9}$$

Here, $\boldsymbol{T} = \{T_1, T_2, T_3, M_1, M_2, M_3\}^{\mathrm{t}}$ is the vector of internal forces and moments and D is the matrix of elastic stiffnesses. We assume the shell to be structurally anisotropic and apply the Kirchhoff–Love hypothesis. In this case, the matrix D has the form

$$D = \begin{bmatrix} b_{11} & b_{12} & 0 & c_{11} & c_{12} & 0 \\ b_{12} & b_{22} & 0 & c_{12} & c_{22} & 0 \\ 0 & 0 & b_{33} & 0 & 0 & c_{33} \\ c_{11} & c_{12} & 0 & d_{11} & d_{12} & 0 \\ c_{12} & c_{22} & 0 & d_{12} & d_{22} & 0 \\ 0 & 0 & c_{33} & 0 & 0 & d_{33} \end{bmatrix},$$

where the elements b_{ij}, c_{ij}, and d_{ij} are determined in accordance with [1].

Let us consider the shell FEs subjected to the action of a system of a nonuniform surface load $\boldsymbol{q} = \{q_1, q_2, q_3\}^{\mathrm{t}}$, a system of contour forces and moments $\boldsymbol{R}_k = \{P_{1k}, P_{2k}, P_{3k}, M_{1k}, M_{2k}, M_{3k}\}^{\mathrm{t}}$, and a system of local forces and moments $\boldsymbol{R}_l = \{P_{1l}, P_{2l}, P_{3l}, M_{1l}, M_{2l}, M_{3l}\}^{\mathrm{t}}$. The subscripts 1, 2, and 3 correspond to the directions of the x, y, and z axes, respectively. The total potential FE energy is $\Pi = W - V$ (W is the strain energy and V is the work of external forces). According to [1], we have

$$\begin{gathered} W = \frac{1}{2}\iint\limits_s \boldsymbol{T}^{\mathrm{t}}\boldsymbol{e}\,ds = \frac{1}{2}\iint\limits_s (\boldsymbol{T}^{\mathrm{t}}\boldsymbol{e}_l + \boldsymbol{T}^{\mathrm{t}}\boldsymbol{e}_n)\,ds = \frac{1}{2}\iint\limits_s (\boldsymbol{e}_l^{\mathrm{t}}D\boldsymbol{e}_l + \boldsymbol{e}_l^{\mathrm{t}}D\boldsymbol{e}_n + \boldsymbol{e}_n^{\mathrm{t}}D\boldsymbol{e}_l + \boldsymbol{e}_n^{\mathrm{t}}D\boldsymbol{e}_n)\,ds, \\ V = \iint\limits_s \boldsymbol{q}^{\mathrm{t}}\tilde{\boldsymbol{u}}\,ds + \int\limits_{l_k} \boldsymbol{R}_k^{\mathrm{t}}\tilde{\boldsymbol{u}}_k\,dl_k + \boldsymbol{R}_l^{\mathrm{t}}\bar{\boldsymbol{u}}_l. \end{gathered} \tag{10}$$

Let us write the Lagrange variational equation $\delta\Pi = \delta W - \delta V = 0$. Varying Eqs. (10) over the FE node unknowns, we obtain

$$\delta\Pi = \iint\limits_s (\boldsymbol{e}_l^{\mathrm{t}} D\,\delta\boldsymbol{e}_l + \boldsymbol{e}_l^{\mathrm{t}} D\,\delta\boldsymbol{e}_n + \boldsymbol{e}_n^{\mathrm{t}} D\,\delta\boldsymbol{e}_l + \boldsymbol{e}_n^{\mathrm{t}} D\,\delta\boldsymbol{e}_n)\,ds - \delta V,$$

$$\delta V = \iint\limits_s \boldsymbol{q}^{\mathrm{t}}\,\delta\tilde{\boldsymbol{u}}\,ds + \int\limits_{l_k} \boldsymbol{R}_k^{\mathrm{t}}\,\delta\tilde{\boldsymbol{u}}_k\,dl_k + \boldsymbol{R}_l^{\mathrm{t}}\,\delta\bar{\boldsymbol{u}}_l. \tag{11}$$

Substituting Eqs. (7) and (8) into Eqs. (11), we obtain a system of nonlinear algebraic equations for FE node unknowns. Taking into account the compatibility conditions for FE displacements and the boundary conditions, in accordance with [2], we obtain a system of nonlinear algebraic equations for all node unknowns of the shell:

$$K\boldsymbol{u}' - \boldsymbol{Q} = 0. \tag{12}$$

Here, K is the shell stiffness matrix whose elements are obtained by summation of the stiffness matrix elements for individual FEs with the use of the index matrix [3], $\boldsymbol{Q}$ is the vector of the generalized node forces of the shell, whose elements are obtained by summation of vector elements of individual FEs with the use of the index matrix, $\boldsymbol{u}' = \{\bar{\boldsymbol{u}}_1, \ldots, \bar{\boldsymbol{u}}_k, \ldots, \bar{\boldsymbol{u}}_{m\times n}\}^{\mathrm{t}}$ is the vector of the node unknowns of the shell, and $\bar{\boldsymbol{u}}_k$ is the vector of the node displacements of the kth node of the finite-element grid.

System (12) is solved by the marching (over the load) method with the use of the Newton–Kantorovich method of linearization at each step [4, 5]. The equation of this method for the shell can be written as [6]

$$H\,\Delta\boldsymbol{u}' = \boldsymbol{Q} - \boldsymbol{G}, \qquad (\boldsymbol{u}')^{n+1} = (\boldsymbol{u}')^n + \Delta\boldsymbol{u}', \tag{13}$$

where H is the Hesse matrix of the shell, whose elements are the second derivatives of the potential strain energy and $\boldsymbol{G}$ is the gradient of the potential strain energy, whose elements are the first derivatives of the potential strain energy. The system of linear equations (13) is solved by the Kraut method [7] with the use of expansion of the Hesse matrix $H = L^{\mathrm{t}}DL$ into the diagonal and two triangular matrices. The node displacements are used to determine displacements (3), strains (6), and moments and forces (9). The stable state of the shell is verified by checking the positive definiteness of the Hesse matrix by the Silvester criterion [4], which reduces to checking the positiveness of the elements of the diagonal matrix D. The appearance of negative elements corresponds to shell buckling. Calculating the load parameter value at which the equilibrium state is unstable, we can determine the form of shell buckling by solving the system $H\boldsymbol{d} = 0$, where $\boldsymbol{d}$ is the vector of bifurcation node displacements. For this purpose, we have to find the linearly dependent (degenerate) row of the matrix H corresponding to the first negative element of the matrix D. The elements of this row and of the corresponding column of the matrix H are assumed to be equal to zero. A unity is placed instead of the diagonal coefficient, and the corresponding column multiplied by the subcritical displacement corresponding to the degenerate row is transferred to the right side of the system. By solving the thus-obtained system, we find the form of shell buckling. If the strain curves contain a limiting point, the form of shell buckling is determined with allowance for the nonlinear initial stress–strain state for the load close to the limiting value.

Study of Nonlinear Deformation and Stability of a Reinforced Shell. Let us consider the problem of nonlinear deformation and stability of an elliptical cylindrical shell clamped on one side and reinforced by a set of stringers, which is subjected to a bending moment M, twisting moment M_{tw}, transverse force Q, and internal pressure q (see Fig. 1). The loaded edge of the shell is reinforced by a frame with a large stiffness with respect to bending in its plane. We replace the action of the bending moment M by the action of nonuniform axial forces over the shell circumference $T_x = Mz_1/J$ (z_1 is the distance from the shell contour points to the major axis of the ellipse a). The action of the transverse force Q is replaced by the action of running boundary tangential forces $T_{xy} = QS/J$ (S and J are the static moment and the moment of inertia of the shell cross section with respect to the ellipse axis a). The action of the twisting moment M_{tw} is replaced by the action of uniform tangential forces over the shell circumference $T_{xy} = M_{\mathrm{tw}}/(2\omega)$ ($\omega = \pi ab$ is the clear area of the shell cross section). The action of internal pressure is replaced by the action of normal pressure and boundary tensile forces $N = q\omega$.

Let us consider a shell with specified lengths of the semi-axes a and b. The calculations were performed with the following values of parameters: shell length $L = 1000$ mm, thickness $h = 2$ mm, $R_0 = 1900$ mm, Young's modulus $E = 7\cdot 10^4$ MPa, and Poisson's ratio $\nu = 0.3$. The cross-sectional area of the stringers was $F_{\mathrm{str}} = 100\ \mathrm{mm}^2$,

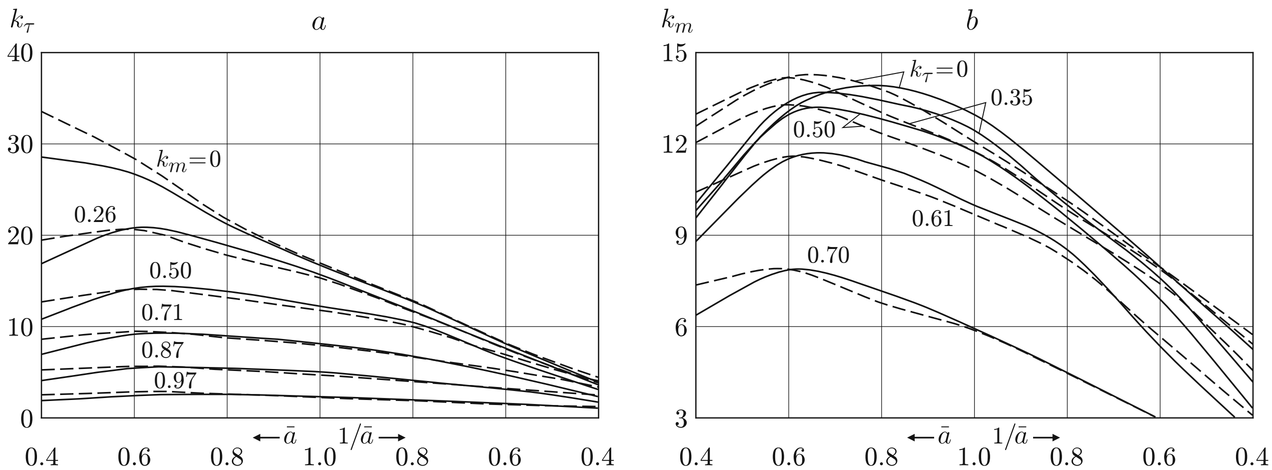

Fig. 2. Parameters k_τ (a) and k_m (b) versus the ellipticity parameter $\bar{a}$ at $q = 1.5$ atm and $\bar{k}_p = 2.5$: the solid and dashed curves show the data corresponding to the nonlinear and linear initial stress–strain state, respectively.

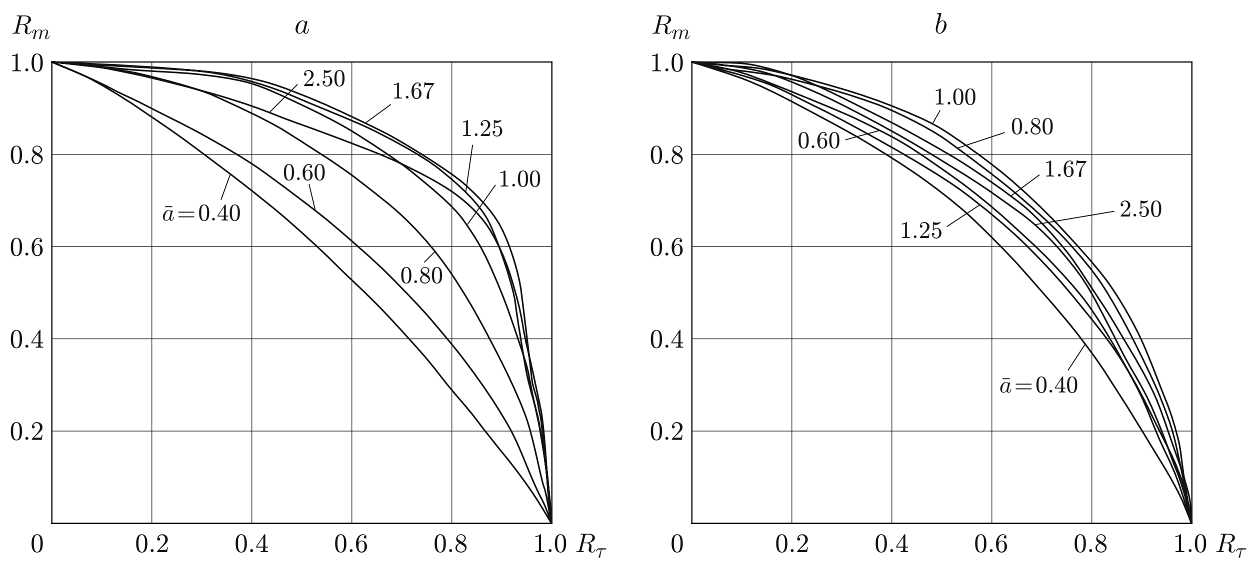

Fig. 3. Dependence $R_m(R_\tau)$ in the case of the linear (a) and nonlinear (b) initial stress–strain states for different values of the parameter $\bar{a}$.

the own moment of inertia was 3333 mm^4, the step between the stringers was 150 mm, and the eccentricity of the stringers with respect to the shell mid-surface was 10 mm.

Let us introduce the following notations: $k_m = M^*/M_0$, $k_p = M^*_{\rm tw}/M_{\rm twist0}$, $k_\tau = Q^*/Q_0$. Here M^*, $M^*_{\rm tw}$, and Q^* are the critical values of the bending moment, twisting moment, and transverse force, $M_0 = \pi E R_0 h^2/\sqrt{3(1-\nu^2)}$, $M_{\rm twist0} = 2\pi C R_0^2 S_b$, and $Q_0 = \pi R_0 C S_b$ are the classical critical values of the bending moment, twisting moment, and transverse force for a non-reinforced circular cylindrical shell of radius R_0 under separate actions of M, $M_{\rm tw}$, Q, and $q = 0$; $S_b = 0.74(Eh/(1-\nu^2)^{5/8})(h/R_0)^{5/4}(R_0/L)^{1/2}$; $C = 0.953$; $R_0 = P/(2\pi) = (2a/\pi)E(\pi/2, b/a)$ is the equiperimeter radius, i.e., the radius of the circular shell with a perimeter P equal to the ellipse perimeter; $E(\pi/2, b/a)$ is the total elliptical integral of the second kind.

Figure 2a shows the dependence of the parameter k_τ on the shell ellipticity parameter $\bar{a} = a/b$ for equiperimeter shells in the cases of the linear and nonlinear initial stress–strain states at $q = 1.5$ atm, $\bar{k}_p = M_{\rm tw}/M_{\rm twist0} = 2.5$, and different values of the parameter k_m. It is seen that the allowance for nonlinearity leads to a decrease (by 13%)

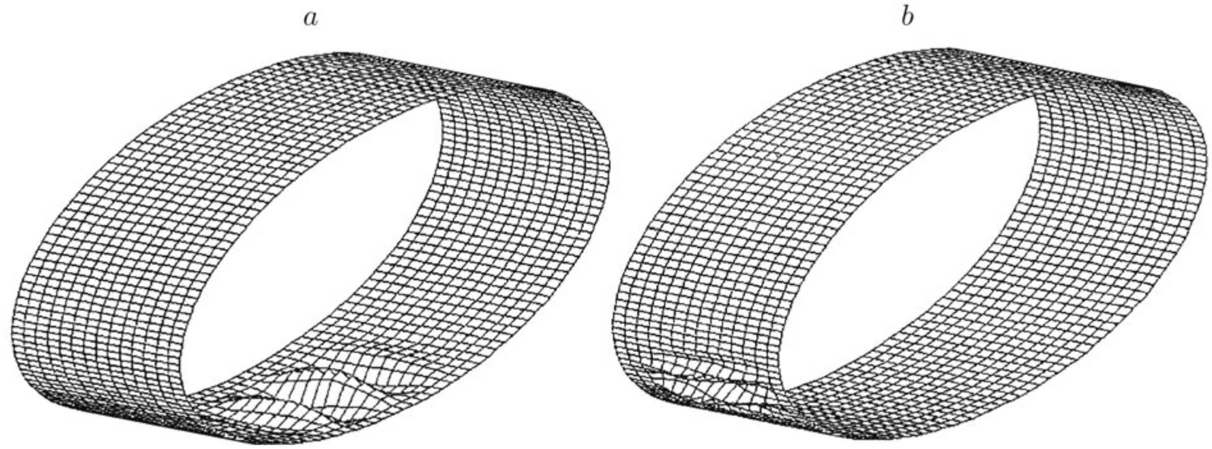

Fig. 4. Forms of shell buckling: (a) buckling in the zone of the maximum tangential forces; (b) buckling in the zone of the maximum compressive forces.

in the critical values of the transverse force for "high" ($\bar{a} < 0.6$) shells and to their insignificant increase (by 5%) at $\bar{a} > 0.6$.

Figure 2b shows the parameter k_m as a function of the shell ellipticity parameter $\bar{a}$ for equiperimeter shells in the cases of the linear and nonlinear initial stress–strain states at $q = 1.5$ atm, $\bar{k}_p = 2.5$, and different values of the parameter k_τ. It follows from Fig. 2b that the allowance for nonlinearity in this case also leads to a decrease in the critical values of the moment (by 22%) for "high" ($\bar{a} < 0.7$) shells and to an insignificant increase in these values (by 5%) for "low" ($\bar{a} > 0.7$) shells.

Figure 3 shows the dependence between the relative critical values of the bending moment R_m and the relative critical values of the transverse force R_τ for equiperimeter shells in the cases of the linear (Fig. 3a) and nonlinear (Fig. 3b) initial stress–strain states for different values of the shell ellipticity parameter $\bar{a}$. It follows from Fig. 3 that the curves of these dependences are convex, and their curvature in the case of elliptical shells is smaller than in the case of circular shells.

Figure 4 shows the typical forms of shell buckling. It is seen that the buckling shape depends substantially on the ratio a/b, the type of the load, and the parameters R_m and R_τ. The shells usually lose their stability in the zone of the maximum tangential forces with formation of two or three folds in the region with a moderate curvature of the shells if the transverse force prevails (Fig. 4a) or in the zone of the maximum compressive forces (in the lower part of the shell) with formation of several transverse waves if the bending moment prevails (Fig. 4b).

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